ON THE DIMENSIONS OF THE IRREDUCIBLE MODULES OF LIE ALGEBRAS OF CLASSICAL TYPE(1)

BY

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Introduction. In this and another paper [4], partial answers are given to two of the questions raised in §II.5 of [3]. Because the results in [4] hold for a more extensive class of Lie algebras of classical type than those considered in this paper, the two questions have been treated separately. Let $\mathfrak{L}$ be a Lie algebra over an algebraically closed field Ω of characteristic $p>7$ whose Killing form is nondegenerate, and let M be an irreducible restricted right $\mathfrak{L}$-module. We shall give a computable sufficient condition on M in order for Weyl's formula [9, p. 359], suitably interpreted, to give the dimension of M over Ω. The problem of calculating the dimension of M in all cases remains unsolved. In the last section we obtain an upper bound for the dimension of any irreducible restricted right $\mathfrak{L}$-module, and from this result it follows that Weyl's formula as we have interpreted it does not always give the dimension of M.

1. **Definitions and preliminary results.** Familiarity with the papers [1; 2] and [3] is assumed. First we list some notations.

Ω algebraically closed field of characteristic $p>7$;

$\mathfrak{L}$ Lie algebra over Ω whose Killing form is nondegenerate;

M irreducible restricted right $\mathfrak{L}$-module;

λ^* maximal weight of M;

$\Delta=\{\alpha_1, \cdots, \alpha_l\}$ a maximal simple system of roots of $\mathfrak{L}$ with respect to a fixed Cartan subalgebra $\mathfrak{H}$ (see [2, p. 97]);

h_i the unique element in $[\mathfrak{L}_{-\alpha_i}\mathfrak{L}_{\alpha_i}]$ such that $\alpha_i(h_i)=2$.

By Corollary 1, p. 107, of [2], Δ is a linearly independent set, and it follows that $h_1, \cdots, h_l$ is a basis of $\mathfrak{H}$, so that λ^* is uniquely determined by the vector $(\lambda^*(h_1), \cdots, \lambda^*(h_l))$ with coefficients in the prime field Ω_0 of Ω. By the main theorem of [2, p. 104], there exists a semi-simple Lie algebra $\mathfrak{L}'$ over the complex field with the following properties. Let $\mathfrak{H}'$ be a Cartan subalgebra of $\mathfrak{L}'$. Then there exists a one-to-one mapping $\alpha\rightarrow\alpha'$ of the set of roots of $\mathfrak{L}$ with respect to $\mathfrak{H}$ onto the set of roots of $\mathfrak{L}'$ with respect to $\mathfrak{H}'$ which is compatible with the operations of addition and subtraction defined on the system of roots. If $\Delta'=\{\alpha_1', \cdots, \alpha_l'\}$ then Δ' is a maximal simple system of roots of $\mathfrak{L}'$. For each root α' of $\mathfrak{L}'$, let $H_{\alpha'}$ be the unique element in $[\mathfrak{L}'_{-\alpha'}, \mathfrak{L}'_{\alpha'}]$ such that $\alpha'(H_{\alpha'})=2$, and let $H_i=H_{\alpha'_i}$, $1\leqq i\leqq l$. We can choose

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$E_{\alpha'} \in \mathfrak{L}'_{\alpha'}$ such that $[E_{-\alpha'}, E_{\alpha'}] = H_{\alpha'}$ for all α', and the elements $E_{\alpha'}$ together with the $H_{i,1\leqq i\leqq l}$, form a basis (X_i) of $\mathfrak{L}'$ such that all the constants of structure of this basis belong to the ring Z_p of p-adic integers (see [2, pp. 102–104]). Let $\mathfrak{L}_0'$ be the Lie ring $\sum Z_p X_i$; then $\mathfrak{L}_0'/p\mathfrak{L}_0'$ is a Lie algebra over Ω_0 which becomes isomorphic to $\mathfrak{L}$ upon extension of the base field from Ω_0 to Ω.

Let $\mathfrak{F}$ be the set of *integral linear functions* on $\mathfrak{H}'$ i.e. $\mathfrak{F}$ consists of those linear functions λ such that $\lambda(H_i)$ is a rational integer for $1 \leqq i \leqq l$. The dominant integral functions are those such that $\lambda(H_i) \geqq 0$, $1 \leqq i \leqq l$. The elements of $\mathfrak{F}$ span a vector space over the field of rational numbers, and this vector space carries a scalar product whose value at the pair λ, μ in $\mathfrak{F}$ is given as follows. Let $B(X, Y)$ be the Killing form on $\mathfrak{L}'$, and for each $\lambda \in \mathfrak{F}$ let J_λ be the unique element in $\mathfrak{H}'$ such that $\lambda(H) = B(J_\lambda, H)$ for all H in $\mathfrak{H}'$. The value of the scalar product of λ and μ is given by $\langle\lambda, \mu\rangle = B(J_\lambda, J_\mu)$, and it is known that $\langle\lambda, \mu\rangle$ is a rational number for λ, μ in $\mathfrak{F}$. Because in this case the Killing form is nondegenerate modulo p we shall prove that we have

$$\langle\lambda, \mu\rangle \in Z_p, \qquad \lambda, \mu \in \mathfrak{F}. \tag{1}$$

Let $\lambda \in \mathfrak{F}$; then $\lambda = \sum_{i=1}^{l} q_i \alpha_i'$ where the q_i are rational numbers. Consider the system of equations

$$\lambda(H_j) = \sum_{i=1}^{l} q_i \alpha_i'(H_j), \qquad j = 1, 2, \cdots, l.$$

The numbers $\lambda(H_j)$ belong to Z_p, and the $\alpha_i'(H_j)$ are elements of Z_p such that $\det(\alpha_i'(H_j)) \not\equiv 0 \pmod{p}$ (see [1, p. 164]). It follows that the $q_i \in Z_p$ for all i, and since $\langle\alpha_i', \alpha_j'\rangle \in Z_p$(2), $1 \leqq i, j \leqq l$, we have proved (1).

Now let $\{H_i\}$ and $\{K_i\}$ be dual bases of $\mathfrak{H}'$ with respect to the Killing form, and choose elements $F_{\alpha'} \in \mathfrak{L}'$ such that $[F_{-\alpha'} F_{\alpha'}] = J_{\alpha'}$ for each root $\alpha' \neq 0$. We then form the *Casimir element*

$$\Gamma = \sum_{\alpha' \neq 0} F_{-\alpha'} F_{\alpha'} + \sum_{i=1}^{l} H_i K_i,$$

which is known to belong to the center of the universal associative algebra $U(\mathfrak{L}')$ of $\mathfrak{L}'$. For each irreducible finite dimensional right $\mathfrak{L}'$-module V with maximal weight Λ, we have

$$v\Gamma = \gamma(\Lambda)v \qquad v \in V,$$

where $\gamma(\Lambda)$ is a fixed nonnegative rational number which is given explicitly by

$$\gamma(\Lambda) = \langle\Lambda, \Lambda\rangle + \sum_{\alpha'>0} \langle\Lambda, \alpha'\rangle = \langle\Lambda, \Lambda + 2\rho\rangle \tag{2}$$

(2) To prove this assertion, we begin with the known result that $2\langle\alpha_i', \alpha_j'\rangle\langle\alpha_j', \alpha_j'\rangle^{-1}$ is a rational integer for all i and j. Then by the argument in the proof of Proposition 6, p. 104 of [2], we know that $\langle\alpha_j', \alpha_j'\rangle = B(J_{\alpha'_j}, J_{\alpha'_j})$ is a unit in Z_p, and hence $\langle\alpha_i', \alpha_j'\rangle \in Z_p$ as required.

where ρ is one half the sum of the positive roots of $\mathfrak{L}'$ with respect to $\mathfrak{H}'$. The number $\gamma(\lambda)$ is defined by (2) for every dominant integral function λ, and is by (1) an element of Z_p. In particular if λ is a dominant integral function which is a weight of V, then it is known [5, p. 373] that

$$\gamma(\lambda) < \gamma(\Lambda), \qquad \lambda \neq \Lambda. \tag{3}$$

Any $\mathfrak{L}'$-module V is also a completely reducible module for the three dimensional simple subalgebra $\mathfrak{U}_{\alpha'}$ of $\mathfrak{L}'$ with basis $E_{\alpha'}$, $E_{-\alpha'}$, $H_{\alpha'}$, where α' is any nonzero root. This fact together with the explicit determination of the irreducible $\mathfrak{U}_{\alpha'}$-modules will be used frequently throughout the paper to obtain information about the linear transformations on V induced by $E_{\alpha'}$ and $E_{\alpha'}E_{-\alpha'}$. For any $\alpha' \neq 0$, the irreducible modules of $\mathfrak{U}_{\alpha'}$ are described as follows. There exists one and only one module of dimension k for each $k=1, 2, \cdots$, and if $[v_0, v_1, \cdots, v_{k-1}]$ is a basis for this module, the action of $\mathfrak{U}_{\alpha'}$ is given by

$$\begin{aligned} v_iE_{-\alpha'} &= v_{i+1}, \ 0 \leqq i \leqq k-2; \qquad v_{k-1}E_{-\alpha'} = 0; \\ v_iE_{\alpha'} &= i(k-i)v_{i-1}, \qquad 0 \leqq i \leqq k-1; \\ v_iH_{\alpha'} &= (k-1-2i)v_i, \qquad 0 \leqq i \leqq k-1. \end{aligned} \tag{4}$$

From these formulas it is immediate that $VE_{\alpha'}^m=0$ for all roots $\alpha' \neq 0$ if and only if

$$l(V) = \max_{\alpha',\lambda} |\lambda(H_{\alpha'})| < m,$$

where the maximum is taken over all roots $\alpha' \neq 0$ and all weight λ of V (see [6, p. 306]).

All these preparations are brought to bear on the irreducible $\mathfrak{L}$-module M by means of the following definition.

Definition. Let Λ be the dominant integral function such that $\Lambda(H_i)$ is the integer, $0 \leqq \Lambda(H_i) < p$, whose residue modulo p is equal to $\lambda^*(h_i)$, $1 \leqq i \leqq l$. Let V be the irreducible $\mathfrak{L}'$-module whose maximal weight is Λ. Then V is called the *associated module* of M.

Now we can state the principal result of the paper.

Theorem 1. *Let M be an irreducible restricted $\mathfrak{L}$-module whose associated module V satisfies the following conditions*: (i) $l(V) < p$; *and* (ii) *for any dominant weight λ different from the maximal weight Λ of V*, $\gamma(\lambda) \not\equiv \gamma(\Lambda)$ (mod p). *Then we have* $\dim M = \dim V$, *and this dimension is given by Weyl's formula*

$$\dim M = \dim V = \prod_{\alpha'>0} \frac{(\Lambda+\rho)(H_{\alpha'})}{\rho(H_{\alpha'})}. \tag{5}$$

2. **Proof of Theorem** 1. We remark first that in view of (3) the hypothesis (ii) amounts to the assumption that the inequalities (3) cannot be replaced

by congruence modulo p. We point out also that it is sufficient to prove that $\dim M = \dim V$, since the formula (5) is known to give the dimension of V.

We begin with an observation about modules over Z_p. Let Q be a free Z_p-module with basis $x_1, \cdots, x_n$. A set of vectors $y_1, \cdots, y_m$ is called a *p-independent set* provided that $\sum a_i y_i \in pQ$ implies $a_i \in pZ_p$ for any set of a_i from Z_p. We shall prove that *a set* $y_1, \cdots, y_n$ *of vectors in* Q *is a* Z_p*-basis if the* y_i *are p-independent.* Let $y_i = \sum a_{ij} x_j$, $a_{ij} \in Z_p$. Because the y_i's are p-independent, we have $\det(a_{ij}) \not\equiv 0 \pmod{p}$. Since an element of Z_p not divisible by p is a unit, the matrix (a_{ij}) is invertible in the set of n by n matrices with coefficients in Z_p. From this it follows that $y_1, \cdots, y_n$ is a Z_p-basis of Q.

Now let V be the associated module of M, and let v be a maximal vector in V. Let V_0 be the smallest Z_p-submodule of V containing v and invariant relative to $\mathfrak{L}_0'$. Then V_0 is a finitely generated free Z_p-module which spans V over the complex field, and by the argument on p. 170 of [1], V_0 is generated by vectors of the form $vE_{\gamma'_1} \cdots E_{\gamma'_s}$, $s \geqq 0$, where the γ_i' are negative roots. Because these vectors are all weight vectors it follows that

$$V_0 = \sum_{\lambda} \oplus (V_0 \cap V_\lambda) \tag{6}$$

where the sum is taken over the weights λ of V. From (6) it is clear that V_0 has a Z_p-basis consisting of weight vectors.

LEMMA 1. *Suppose that* $l(V) < p$, *and let* $\alpha_i \in \Delta$. *Then* V_0 *has a* Z_p*-basis* $(v_j^{(i)}) 1 \leqq j \leqq \dim V$ *of weight vectors* $v_j^{(i)}$ *which are proper vectors of* $E_{\alpha'_i} E_{-\alpha'_i}$ *with proper values which are either zero or units in* Z_p.

Proof. We begin with some known facts about representation of the three dimensional simple Lie algebra $\mathfrak{U}_{\alpha'_i}$ which are more or less immediate consequences of the formulas (4). Let $v \in V_\lambda$ be a weight vector such that $vE_{\alpha'_i} = 0$. Then the vectors $\{vE^k_{-\alpha'_i}\}$, $k \geqq 0$, generate an irreducible $\mathfrak{U}_{\alpha'_i}$-submodule of V. Because $0 \leqq \lambda(H_i) \leqq l(V) < p$, $vE^p_{-\alpha'_i} = 0$ and the nonzero vectors $vE^k_{-\alpha'_i}$ are weight vectors, and are proper vectors of $E_{\alpha'_i} E_{-\alpha'_i}$ whose proper values are either zero or units in Z_p. Now let N^i be the null space of $E_{\alpha'_i}$ in V, and let $N_0^i = N^i \cap V_0$. We prove first that

$$N_0^i = \sum_{\lambda} \oplus (N^i \cap V_\lambda \cap V_0). \tag{7}$$

It is sufficient to prove that N_0^i is contained in the sum on the right hand side. Let $v \in N_0^i$; then by (6), $v = \sum v_\lambda$ where the $v_\lambda \in V_\lambda \cap V_0$, and the λ's are distinct weights. Applying $E_{\alpha'_i}$ we have

$$0 = vE_{\alpha'_i} = \sum_{\lambda} v_\lambda E_{\alpha'_i},$$

and since the $v_\lambda E_{\alpha'_i}$ belong to the distinct weights $\lambda + \alpha_i'$, we have $v_\lambda E_{\alpha'_i} = 0$ for all λ, and the $v_\lambda \in N^i \cap V_\lambda \cap V_0$ as required.

Because of (7) we can find a Z_p-basis of N_0^i consisting of weight vectors w_j; then the w_j span N^i over the complex field. From the formulas (4) and the fact that V is a completely reducible $\mathfrak{U}_{\alpha'_i}$-module, it follows that the nonzero vectors $w_jE^k_{-\alpha'_i}$ are linearly independent and span V over the complex field. These vectors satisfy all the requirements of the lemma if we can prove that they are p-independent, for then by the remark preceding Lemma 1, they will form a Z_p-basis of V_0. Let

$$w = \sum a_{jk}w_jE^k_{-\alpha'_i} \in pV_0, \qquad a_{jk} \in Z_p.$$

Write w in the form

$$w = \sum a_{jN}w_jE^N_{-\alpha'_i} + \sum a_{j,N-1}w_jE^{N-1}_{-\alpha'_i} + \cdots;$$

then from the formulas (4) we have

$$wE^N_{\alpha'_i} = \sum a_{jN}\zeta_{jN}w_j \in pV_0,$$

where the ζ_{jN} are units in Z_p. Because the w_j are p-independent we have $a_{jN}\in pZ_p$, and the terms $\sum a_{jN}w_jE^N_{-\alpha_i}$ can be dropped from the expression for w. Continuing in this way we see that the $w_jE^k_{-\alpha'_i}$ are p-independent, and Lemma 1 is proved.

LEMMA 2. *Let M be an irreducible restricted $\mathfrak{L}$-module with maximal weight λ^*, and let V be the associated module M. Then $\overline{V}=(V_0/pV_0)^\Omega$ is a (not necessarily restricted) $\mathfrak{L}$-module, and $\overline{V}$ has a restricted homomorphic image R which is $\mathfrak{L}$-isomorphic to M*(3).

Proof. We introduce first some notation which will be used here and later in the proof of the theorem. Let $X\rightarrow X^*$ be the natural mapping of $\mathfrak{L}_0'$ $\rightarrow\mathfrak{L}_0'/p\mathfrak{L}_0'$. Because $(\mathfrak{L}_0'/p\mathfrak{L}_0')^\Omega=\mathfrak{L}$, we may regard the X^* as elements of $\mathfrak{L}$ which span $\mathfrak{L}$ over Ω. For any $\eta\in Z_p$ we let η^* be the image of η under the natural map of $Z_p\rightarrow\Omega_0$. The elements H_i^*, $1\leqq i\leqq l$ form a basis of $\mathfrak{H}$, and we may assume that $\mathfrak{L}_\alpha=\Omega E^*_{\alpha'}$ for all roots $\alpha\neq 0$. We shall denote the image of v in V_0 under the natural map of $V_0\rightarrow V_0/pV_0$ by $[v]$. $\overline{V}$ becomes a right $\mathfrak{L}$-module if we define

$$a^*[v] = [av], \qquad a \in Z_p, \qquad v \in V_0,$$

and

$$[v]X^* = [vX], \qquad v \in V_0, \qquad X \in \mathfrak{L}_0'.$$

If Λ is the maximal weight of V and v_0 the maximal vector in V_0 which generates V_0, then $[v_0]$ is a maximal vector in $\overline{V}$ of weight λ^*. Let S be a maximal $\mathfrak{L}$-submodule of $\overline{V}$ which does not contain $[v_0]$; then $R=\overline{V}/S$ is an irreducible

(3) This lemma is an improvement of Theorem 6 of [1].

$\mathfrak{L}$-module with maximal weight λ^*. If we can show that R is a restricted $\mathfrak{L}$-module, then $R \cong M$ by Theorem 1 of [3]. It is sufficient to prove that $R(E^*_{\alpha'})^p = 0$ for all roots α' and $R[(H^*_i)^p - H^*_i] = 0$. In any case $(E^*_{\alpha'})^p$ belongs to the centralizer of R, so that by Schur's Lemma $(E^*_{\alpha'})^p = \xi \cdot 1$ for some $\xi \in \Omega$. On the other hand, $E_{\alpha'}$ is nilpotent on V, hence $E^*_{\alpha'}$ is nilpotent on R, and we have $\xi = 0$. In order to discuss $(H^*_i)^p - H^*_i$, we recall that $\overline{V}$ is spanned by the elements $\bar{v} = [v_0]E^*_{\gamma'_1}, \cdots, E^*_{\gamma'_s}$, and for such an element we have

$$\begin{aligned}\bar{v}H^*_i &= [v_0]E^*_{\gamma'_1}, \cdots, E^*_{\gamma'_s}H^*_i = [v_0E_{\gamma'_1}, \cdots, E_{\gamma'_s}H_i] \\ &= [\mu(H_i)v_0] = \mu(H_i)^*v_0\end{aligned}$$

where $\mu = \Lambda + \gamma'_1 + \cdots + \gamma'_s$ and Λ is the maximal weight of V. Since $\mu(H_i)^* \in \Omega_0$, $\bar{v}(H^*_i)^p - \bar{v}H^*_i = 0$, and we have proved that R is restricted. This completes the proof of Lemma 2.

Now consider the Casimir element Γ. Since we can express $F_{\alpha'} = 2^{-1}\langle\alpha', \alpha'\rangle E_{\alpha'}, \alpha' > 0, F_{-\alpha'} = E_{-\alpha'}, \alpha' > 0$, and solve the equations $K_i = \sum b_{ij}H_j$ for $b_{ij} \in Z_p$ because $B(K_i, H_j) = \delta_{ij}, \cdots$, we can assume that the elements $F_{\alpha'}$, H_i, and K_i all belong to $\mathfrak{L}'_0$. Therefore $\Gamma^* = \sum F^*_{-\alpha'}F^*_{\alpha'} + \sum H^*_iK^*_i$ is an element of the center of the universal associative algebra $U(\mathfrak{L})$ of $\mathfrak{L}$.

Lemma 3. *Let Q be a restricted right $\mathfrak{L}$-module and let N be an irreducible composition factor of Q. Suppose that $\Gamma^* = \delta \cdot 1$ on Q for some $\delta \in \Omega$. Then $\delta = \gamma(\mu)^*$, where μ is the maximal weight of the associated module W of N.*

Proof. On W we have $\Gamma = \gamma(\mu) \cdot 1$. Therefore on $\overline{W} = (W_0/pW_0)^\Omega$ we have $\Gamma^* = \gamma(\mu)^* \cdot 1$. By Lemma 2, $\overline{W}$ has a homomorphic image isomorphic to N, and it follows at once from this that $\delta = \gamma(\mu)^*$.

Finally we come to the proof of the theorem. Because $l(V) < p$, we have $V_0(E^*_{\alpha'})^p = 0$ for all roots α', and it follows that $\overline{V}$ is a restricted $\mathfrak{L}$-module. We suppose the theorem is false; then $\overline{V}$ contains a proper submodule T. The enveloping algebra of the transformations $E^*_{\alpha'}$, $\alpha' > 0$, on T is a nilpotent algebra, so that T contains a maximal vector $[w] \neq 0$. We may assume that $[w]$ is a weight vector. Then we can express $w = A + B + \cdots$, where $A \in V_0 \cap V_{\lambda_A}$, $B \in V_0 \cap V_{\lambda_B}, \cdots$, and where $\lambda_A(H_i) \equiv \lambda_B(H_i) \pmod p$ for $i = 1, \cdots, l$. We may assume also that none of the components $A, B, \cdots$ of w lie in pV_0. For $\alpha_i \in \Delta$, let $\{v_j^{(i)}\}$ be the Z_p-basis of V_0 chosen according to Lemma 1, and let $A = \sum a_j v_j^{(i)}$, for example, where the $v_j^{(i)}$ all have weight λ_A. Because $[w]$ is a maximal vector in $\overline{V}$, we have $[w]E^*_{\alpha'_i} = [wE_{\alpha'_i}] = 0$, and hence $AE_{\alpha'_i}E_{-\alpha'_i} = \sum a_j\mu_j v_j^{(i)} \in pV_0$, where the μ_j are either zero or units in Z_p. Therefore $a_j\mu_j \in pZ_p$, and if $\mu_j \neq 0$, then $a_j \in pZ_p$. Since $A \notin pV_0$, at least one $\mu_j = 0$. By Lemma 1 and the formulas (4), the basis vectors $v_j^{(i)}$ for which $v_j^{(i)}E_{\alpha'_i}E_{-\alpha'_i} = 0$ are among those for which $v_j^{(i)}H_i = \lambda v_j^{(i)}$ for a non-negative λ, and it follows that $\lambda_A(H_i) \geqq 0$. Similarly $\lambda_B(H_i) \geqq 0$, etc. and the weights λ_A, λ_B are dominant weights of V. Because $l(V) < p$, no two dominant weights are congruent modulo p, and it follows that we may assume that $w = A \in V_{\lambda_A}$, where λ_A is a dominant weight different from Λ. If λ' is the weight of $[w]$ then T has a restricted irreducible composition factor N whose maximal weight is λ'. Then λ_A is the maximal weight of the associated module of N. On $\overline{V}$ we have $\Gamma^* = \gamma(\Lambda)^* \cdot 1$, and applying Lemma 3 we obtain $\gamma(\Lambda)^* = \gamma(\lambda_A)^*$. This contradicts the hypothesis (ii) of the theorem. Therefore $\overline{V}$ is irreducible, restricted, and is isomorphic to M by Theorem 1 of [3]. Thus dim M = dim V and Theorem 1 is proved.

3. **Remarks and examples.** We begin with the following estimate of dim M for any irreducible module M.

THEOREM 2. *Let M be an irreducible restricted $\mathfrak{L}$-module and let x_0 be a maximal vector of M. For each positive root α, let m_α be the integer such that $x_0 e_{-\alpha}^{m_\alpha} \neq 0$, $x_0 e_{-\alpha}^{m_\alpha+1} = 0$. Then*

$$\dim M \leqq \prod_{\alpha>0} (m_\alpha + 1). \tag{8}$$

Proof. Let A be the universal associative algebra of $\mathfrak{L}$ regarded as an ordinary Lie algebra (i.e. not taking account of the p-power operation). Then M is an irreducible A-module, and the argument given by Harish-Chandra [7, p. 52] can be applied almost verbatim to prove (8).

Now let V be the associated module of M, and let Λ be the maximal weight of V. For each $\alpha' > 0$, let m'_α be the non-negative integer $< p$ such that $\Lambda(H_{\alpha'}) \equiv m'_\alpha \pmod p$. Then it is easily proved that $m_\alpha = m'_\alpha$.

Now let $\mathfrak{L}'$ be the simple Lie algebra of type A_2 defined over the complex field, and let $H_{\alpha'_1}$, $H_{\alpha'_2}$ be the basis of the Cartan subalgebra of $\mathfrak{L}'$ corresponding to the roots α'_1, α'_2 in a maximal simple system of roots, where $\alpha'_1(H_{\alpha'_1}) = \alpha'_2(H_{\alpha'_2}) = 2$. Then $H_{\alpha'_1+\alpha'_2} = H_{\alpha'_1} + H_{\alpha'_2}$, and it follows that if Λ is a dominant integral function such that $\Lambda(H_{\alpha'_i}) = a_i$, $i = 1, 2$, then the dimension of the irreducible $\mathfrak{L}'$-module V whose maximal weight is Λ is given by

$$\dim V = \prod_{\alpha'>0} \frac{(\Lambda + \rho)(H_{\alpha'})}{\rho(H_{\alpha'})} = \frac{1}{2}(a_1 + 1)(a_2 + 1)(a_1 + a_2 + 2), \tag{9}$$

where $\rho = (\sum_{\alpha'>0} \alpha')/2$ (see also [9, p. 289]).

If $p \neq 3$, then the Killing form of $\mathfrak{L}'$ is nondegenerate modulo p, and it is easily shown by means of (8) and (9) that for any $p > 7$ (so that the general theory is applicable) there exist irreducible restricted $\mathfrak{L}$-modules M with associated modules V such that dim $M \neq$ dim V. In fact, it is enough to choose Λ such that $a_1 + a_2 = p$, $a_i > 0$, $i = 1, 2$. Let V be the irreducible module whose maximal weight is Λ, and M the irreducible restricted $(\mathfrak{L}'_0/p\mathfrak{L}'_0)^\Omega$ module whose maximal weight is λ^*, where $\lambda^*(H_i^*) = a_i^*$, $i = 1, 2$. Then since $H^*_{\alpha'_1+\alpha'_2} = H^*_{\alpha'_1} + H^*_{\alpha'_2}$, we have $\lambda^*(H^*_{\alpha'_1+\alpha'_2}) = a_1^* + a_2^* = 0$, and by (8)

$$\dim M \leqq (a_1 + 1)(a_2 + 1) < \dim V.$$

It is also worthwhile pointing out where our argument in the proof of Theorem 1 fails to apply to arbitrary Lie algebras of classical type in the sense of [8]. For the simple Lie algebra $\mathfrak{L}$ of classical type of class A_n with $p \mid n+1$, the Killing form of $\mathfrak{L}$ is degenerate. In this case $\mathfrak{L}$ is not a modular Lie algebra coming from a complex simple Lie algebra of the same type. Therefore the apparatus of the associated modules and the Casimir element Γ^* as we have used it does not seem to be available.

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